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Higher Approximations to the Far Viscous-Wake Solution

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SUMMARY

Higher order approximations to the laminar boundary-layer flow of an incompressible fluid in the wake of a symmetrical disturbance are given in the present paper. The two-dimensional case, documented elsewhere in great detail, [1], [2], [3] and [4] is reconsidered. The Euler transformation is introduced and higher-order expansion terms are derived. The asymptotic expansions given in the paper are, of course, valid only to the extent that the boundary layer approximations apply, i.e. (for the rotationally symmetrical case) within a space of revolution with the centre of the wake as axis of symmetry. The terms neglected in the complete equations of motion become of order unity for very small x, where the expansion is not applicable in any case (as ε becomes large), and at very large ζ (respectively η), x being given.

The axisymmetrical case is expanded in a like manner, but in both cases the inner and outer coördinate expansion problem of matching with the near wake, considered by Meksyn [16] and Berger [11] is not treated: this, mainly, because its detailed form would depend on the particular upstream conditions obtaining, a subject which is outside the scope of the present work.

1. Introduction

The flow in the wake of streamlined bodies is of considerable technological interest in applications ranging from aerodynamics to the chemical process industries. It will be shown in what follows that it also has analytical properties of notable interest, mainly because it cannot be reduced to single solutions of the similarity type. It has, therefore, been analysed quite extensively, starting with Goldstein [1]. It was first shown that at some distance downstream from a streamlined disturbance with a plane of symmetry, for instance a flat plate, it will be sufficient to use the boundary-layer approximations to the full equations of flow. One may, furthermore, assume that sufficiently far downstream the velocity-defect, compared to the flow infinitely far away from the obstacle, is small: thus regular asymptotic approximations to this solution might presumably be obtained with the velocity defect as the expansion parameter.

It is naturally not clear *a priori* whether a satisfactory perturbation scheme can be developed in this manner: it has been shown by many investigators, Goldstein [1], Stewartson [2] Crane [3], and Chang [4] amongst others that certain restrictive conditions will have to be satisfied by such a scheme.

In his classical analysis of the "far" wake, Goldstein [1] used an Oseen-type of linearization in order to obtain the fundamental term of the expansion. Thereupon two further correction terms could be determined in a straightforward manner. It was then found, however, that the third term gave rise to a velocity defect which decreased with *algebraical* decay to zero, in a direction normal to the plane of symmetry of the wake. Now, it may be shown conclusively that if the boundary-layer type of flow assumed is to match with an outside potential flow, then this decay must be exponentially rapid, a principle known as the "rapid decay of vorticity", Chang [4]. Of course, in other cases this need not necessarily be so, e.g. Brown and Stewartson [5], Rotem [6], Kuiken and Rotem [7].

It is well known that a perturbation procedure of the type described cannot be proven to be unique, and Goldstein himself suggested the possibility that the unsatisfactory result was a fault of the expansion assumed. Stewartson [2] and Crane [3] showed that the inclusion of a transcendental expansion term forming an eigensolution would fulfill the requirements imposed. That leads to a "switchback effect" discussed by Chang [4] and by Rotem and Wygnanski [8], which in effect shows the progressive appearance of logarithmic terms of ever increasing order as the expansion progresses. Stewartson's proof rests upon the iterative solution of a form of the diffusion equation corresponding to Goldstein's case: this yields the first logarithmic term required explicitly.

Moreover, this introduction of the transcendental term will give rise to an arbitrary constant in the expansion, multiplying the resultant "eigensolution", which can be determined neither from the boundary conditions imposed, nor from some gross, overall conservation properties of the flow. From the nature of the governing equations it may be expected that this unknown constant (and possibly subsequent ones) reflect the influence of the initial velocity profile, far upstream of the weak laminar wake, [2]. Therefore, the value of such unknown constants could be determined (approximately only) if it were possible to join the expansions considered here with a known velocity profile upstream. As this latter is not known, our expansion must terminate at the term with the first unknown constant as far as *numerical* computation is concerned. It might now, of course, be possible to proceed in a straightforward numerical manner with the complete equations of flow, Dennis and Dunwoody [9], Plotkin and Flügge-Lotz [10]. On the other hand, we shall show that the expansion possesses interesting properties, insight into which is lost in a purely numerical integration. That fact makes the present investigation highly worthwhile.

The analyses described above were limited to the two-dimensional case, and the validity of the results was restricted to small values of the perturbation parameter (essentially the velocity defect in the plane of symmetry of the wake). The extension of the methods to axisymmetrical flows was first undertaken by Rotem and Wygnanski [8] and by Berger [11]. The purpose of the present investigation is twofold:

(i) To improve the range of applicability of the expansions by the introduction of the *Euler* modification.

(ii) To extend the treatment to the case of the axisymmetrical wake. Though it will be shown that with this geometrical configuration a constant which must be left arbitrary will occur already in the second term, there is some theoretical interest in the determination of the *form* of the higher approximations which are consistent with both the boundary conditions and the requirement of exponential decay of vorticity.

The salient features of the expansions will be discussed in detail.

2. Analysis

The boundary layer equations for the steady, laminar and isothermal flow of an incompressible Newtonian fluid with no pressure gradient are, in dimensionless notation,

$$\left.\begin{aligned} &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=\frac{1}{r^{\delta}}\frac{\partial}{\partial r}\left(r^{\delta}\frac{\partial u}{\partial r}\right)\\ &\frac{\partial u}{\partial x}+\frac{1}{r^{\delta}}\frac{\partial}{\partial r}(v\cdot r^{\delta})=0\end{aligned}\right\} \tag{2.1}$$

where $x=x_1/l$; $r=r_1(U_{\mathrm{ref}}/(l\nu))^{\frac{1}{2}}$; $u=u_1/U_{\mathrm{ref}}$ and $v=v_1(l/(U_{\mathrm{ref}}\nu))^{\frac{1}{2}}$. The suffix "1" will indicate dimensional coordinates, x_1 and r_1 are orthogonal Cartesian coordinates (x_1 in the direction of the streaming flow), u_1 and v_1 the dimensional velocity components respectively, l is an arbitrary reference length, chosen so as to render the coördinate x of order unity, U_{ref} a reference velocity conveniently chosen to render the velocity at $y=0$ at most of order unity, and ν fluid kinematic viscosity. δ takes only the values of zero (for two-dimensional flow) or unity (for axisymmetrical geometry).

The boundary conditions to be fulfilled by the solution to equation (2.1) are,

$$\left.\begin{array}{llll} x>0 & r=0 & \partial u/\partial r=0 & v=0\\ x\geqslant 0 & r=\infty & u=U & \\ x\to\infty & & u\to U\,. & \end{array}\right\} \tag{2.2}$$

Moreover, the total viscous drag obtained by integrating the momentum deficiency from $r=0$ to $r=\infty$ must be the same for any given value of the x coordinate: thus, expressed in terms of the drag D and a drag coefficient C_D,

$$C_D = \frac{D}{\frac{1}{2}\rho l^{1+\delta} U_{\text{ref}}^2} \tag{2.3}$$

we obtain,

$$C_D = 4\pi^\delta \left(\frac{\nu}{lU_{\text{ref}}}\right)^{(1+\delta)/2} \int_0^\infty u(U-u) r^\delta \, dr = \text{constant}. \tag{2.4}$$

In order to restore homogeneous boundary conditions it is convenient to introduce the velocity defect $\bar{u} = U - u$ into equation (2.1), (2.2) and (2.3). Choosing as reference velocity the free-stream velocity we have,

$$(1-\bar{u})\frac{\partial \bar{u}}{\partial x} + v\frac{\partial \bar{u}}{\partial r} = \frac{1}{r^\delta}\frac{\partial}{\partial r}\left(r^\delta \frac{\partial \bar{u}}{\partial r}\right) \tag{2.5}$$

and

$$\frac{d}{dx}\int_0^\infty \bar{u}(1-\bar{u}) r^\delta \, dr = 0 \qquad (x>0). \tag{2.6}$$

Now, sufficiently far downstream $\bar{u} \ll 1$. A first approximation to the velocity distribution in the laminar wake can therefore be obtained through the solution of the linearized equation (2.5), neglecting $\bar{u}$ versus unity in the first term. Lastly, introducing a Stokes' stream function,

$$\bar{u} = -\frac{1}{r^\delta}\frac{\partial \psi}{\partial r} \qquad v = -\frac{1}{r^\delta}\frac{\partial \psi}{\partial x} \tag{2.7}$$

the continuity equation will be fulfilled identically.

We now propose to obtain higher approximations to the solution for the velocity profile, which will enable the calculation for the case in which $\bar{u}$ is not infinitely small; or alternatively expressed, the higher approximations will enable the extension of the range of validity of the solution to values of the coördinate x which are not infinitely large. This extension of the solution is possible in several ways; (i) transforming equations (2.1) into an integral equation which may be solved iteratively, as done by Stewartson [2]; (ii) using Lighthill's method, as Crane [3] has done for the two-dimensional case; or lastly, expanding the solution in an asymptotic series in an ad-hoc manner, Goldstein [1]. This latter method was chosen here because of its inherent simplicity.

3. The Two-Dimensional Wake

The two-dimensional wake has already been examined by many other investigators as mentioned above. Here we introduce the *Euler* transformation into the expansion parameter in order to improve convergence, and examine the resultant changes. The requisite equations of motion and continuity are obtained when δ is put equal to zero in (2.1) through (2.7). Examination of equations (2.1) through (2.6) for $\bar{u} \ll 1$ immediately reveals that the proper coördinate variable will be obtained by

$$\eta = yx^{-\frac{1}{2}}, \tag{3.1}$$

where r has been replaced by y for convenience.

Conversely, had we specialized our considerations to the region near the *start* of the wake, where $\bar{u}$ is of order unity, a coördinate $\tilde{\eta} = yx^{-\frac{2}{3}}$ would have been appropriate, corresponding to Goldstein's "strong (or "near") wake", near the end of the body giving rise to the wake [12].

We shall now introduce a Stokes stream function and assume an ad-hoc expansion as follows,

$$\psi(x, \eta) = x^{\frac{1}{2}} \sum_{k=1}^{\infty} \gamma_k(\varepsilon) f_k(\eta) \tag{3.2}$$

$$\bar{u} = -\sum_{k=1}^{\infty} \gamma_k(\varepsilon) f_k'(\eta) \qquad \gamma_1(\varepsilon) = \varepsilon \tag{3.3}$$

where the functions $\gamma_k(\varepsilon)$ $(k > 1)$ are for the present not known explicitly. That an expansion of the form stipulated in equation (3.1) can be obtained in a way such that the functions f_k depend upon the similarity variable η only, and not also upon x, has been shown by Chang [4].

We adopt the following definitions, and normalization of the drag integral,

$$\varepsilon = Ax^{-\frac{1}{2}} \tag{3.4}$$

$$A = \frac{C_D}{4\sqrt{\pi}} \cdot \left(\frac{U_{\mathrm{ref}\,l}}{\nu}\right)^{\frac{1}{2}} \tag{3.5}$$

$$\int_0^{\infty} f_1'(\eta) d\eta = -\sqrt{\pi} \tag{3.6}$$

and the boundary conditions,

$$f(0) = f''(0) = 0 \tag{3.7}$$

$$f'(\infty) = 0 \,. \tag{3.8}$$

The process of finding the fundamental term consists of inserting,

$$\gamma_1(\varepsilon) = \varepsilon \tag{3.9}$$

into equation (3.1), and considering only the first term. Then,

$$\psi_1 = Af_1(\eta) \tag{3.10}$$

respectively,

$$\bar{u}_1(x, \eta) = -\frac{A}{\sqrt{x}} f_1'(\eta) \,. \tag{3.11}$$

Then the linearized differential equation for the unknown function $f_1(\eta)$ reduces to,

$$f_1''' + \frac{\eta}{2} f_1'' + \tfrac{1}{2} f_1' = 0 \,. \tag{3.12}$$

The solution fulfilling the requisite boundary conditions and normalization conditions (3.8), (3.7) and (3.6) is,

$$f_1' = -\exp[-(\eta/2)^2] \tag{3.13}$$

$$f_1 = -\sqrt{\pi}\, \mathrm{erf}\,(\eta/2) \,. \tag{3.14}$$

The drag-integral condition, equation (3.6), respectively (2.6) now reduces to the requirement that there be no contribution to the constant of order higher than the first in ε.

The form of the functions γ_k cannot be known *a priori*, except for γ_1. It is, of course, well known that an asymptotic expansion such as given in equation (3.2) cannot be proven to be unique. For the two-dimensional viscous wake Stewartson [2] and others [3], [4] have proven that some of the functions $\gamma_k(\varepsilon)$ are necessarily of logarithmic order in the expansion parameter ε. The logarithmic terms ensure an exponentially rapid decay of vorticity with distance normal to the plane of symmetry of the wake, a necessary requirement for that case, in order to match the given outer flow.

Van Dyke [13] has shown that it is permissible to introduce terms transcendental in the expansion parameter ε into a sequence such as (3.2) *ab initio*. Terms which should not appear will then usually give rise to an associated function f_k which vanishes identically. It is on this basis that terms logarithmic in ε were introduced here. It is found, in particular, that in the expansion considered logarithmic terms are necessary in order to ensure the fulfillment of the boundary and drag conditions by a subsequent term of integral power in ε. An additional refinement introduced into the actual expansion here, which will improve on the rate of convergence, is the replacement of the perturbation parameter ε through the *Euler* transformation as follows:

$$\hat{\varepsilon}=\varepsilon/(1+\varepsilon) \tag{3.15}$$

whence

$$\varepsilon=\hat{\varepsilon}+(\hat{\varepsilon})^2+(\hat{\varepsilon})^3+\ldots\,. \tag{3.16}$$

The expansion for the stream function is found after some manipulation as follows (see c.f. Cole [14]),

$$\psi=x^{\frac{1}{2}}\cdot[\hat{\varepsilon}f_1+(\hat{\varepsilon})^2f_2+(\hat{\varepsilon})^3\ln\hat{\varepsilon}f_3+(\hat{\varepsilon})^3f_4+0|(\hat{\varepsilon})^4\ln^2\hat{\varepsilon}|]\,. \tag{3.17}$$

Proceeding as in the case described above the following set of equations for the $f_k(\eta)$ is obtained for $k>1$,

$$f_2'''+\frac{\eta}{2}f_2''+f_2'=\tfrac{1}{2}(f_1'+1)f_1' \tag{3.18}$$

$$f_3'''+\frac{\eta}{2}f_3''+\tfrac{3}{2}f_3'=0 \tag{3.19}$$

$$f_4'''+\frac{\eta}{2}f_4''+\tfrac{3}{2}f_4'=\tfrac{1}{2}[-f_3'+(3f_1'+2)f_2'-f_1''f_2+f_1f_1''-(f_1')^2]\,. \tag{3.20}$$

These equations give rise to the corresponding solutions,

$$f_2'=-e^{-(\eta/2)^2}\cdot[1+\tfrac{1}{2}\cdot(e^{-(\eta/2)^2}+\frac{\sqrt{\pi}}{2}\eta\cdot\operatorname{erf}(\eta/2))] \tag{3.21}$$

$$f_2=-\frac{\sqrt{\pi}}{2}[(2-\varepsilon^{-(\eta/2)^2})\cdot\operatorname{erf}(\eta/2)+\sqrt{2}\operatorname{erf}(\eta/\sqrt{2})] \tag{3.22}$$

$$f_3'=-G_1(1-\eta^2/2)\cdot\exp[-(\eta/2)^2] \tag{3.23}$$

$$f_3=-G_1\eta\cdot\exp[-(\eta/2)^2] \tag{3.24}$$

$$\begin{aligned}f_4'=&+\eta\frac{\sqrt{\pi}}{2}\cdot\left[\frac{1}{4\sqrt{3}}\operatorname{erf}\left(\sqrt{3}\,\frac{\eta}{2}\right)+G_1\cdot\operatorname{erf}\left(\frac{\eta}{2}\right)\right]-e^{-(\eta/2)^2}\cdot\\&\cdot\left\{\frac{\sqrt{\pi}}{2}(\eta^2-2)\cdot\int e^{(\eta/2)^2}\cdot\left[G_1\cdot\operatorname{erf}\left(\frac{\eta}{2}\right)+\frac{1}{4\sqrt{3}}\operatorname{erf}\left(\sqrt{3}\,\frac{\eta}{2}\right)\right]d\left(\frac{\eta}{2}\right)+G_2(2-\eta^2)+\right.\\&+1+G_1+e^{-(\eta/2)^2}(1+\tfrac{1}{2}e^{-(\eta/2)^2})+\tfrac{1}{2}\sqrt{\frac{\pi}{2}}\,\eta\operatorname{erf}\left(\sqrt{2}\,\frac{\eta}{2}\right)-\frac{\pi}{32}(2-\eta^2)\cdot\operatorname{erf}^2(\eta/2)+\\&\left.+\frac{\sqrt{\pi}}{2}\eta(1+\tfrac{1}{4}e^{-(\eta/2)^2})\cdot\operatorname{erf}(\eta/2)\right\}.\end{aligned} \tag{3.25}$$

Here the first equation is identical with that given by Goldstein [1]. The effect of the *Euler* transformation appears only starting at the second term. The constant G_1 was first determined by Stewartson from the condition of exponentially rapid decay of vorticity and is found to be equal to $-1/(4\sqrt{3})$. The constant G_2 must remain indeterminate.

The expressions for the velocity components may now be written down. Thus,

$$\bar{u}=\hat{\varepsilon}\cdot\exp\left[-\left(\frac{\eta}{2}\right)^2\right]\cdot\left\{1+\hat{\varepsilon}\left[1+\tfrac{1}{2}(e^{-(\eta/2)^2}+\frac{\sqrt{\pi}}{2}\eta\,\mathrm{erf}\left(\frac{\eta}{2}\right)\right]+(\hat{\varepsilon})^2\ln\hat{\varepsilon}\cdot\right.$$

$$\cdot\frac{1}{4\sqrt{3}}\left(\frac{\eta^2}{2}-1\right)+(\hat{\varepsilon})^2\cdot\left\{\frac{\sqrt{\pi}}{8\sqrt{3}}(2-\eta^2)\cdot\int e^{(\eta/2)^2}\cdot\left[\mathrm{erf}\left(\frac{\eta}{2}\right)-\mathrm{erf}\left(\sqrt{3}\,\frac{\eta}{2}\right)\right]d\left(\frac{\eta}{2}\right)+\right.$$

$$+1-\frac{1}{4\sqrt{3}}+G_2\cdot(2-\eta^2)+e^{-(\eta/2)^2}(1+\tfrac{1}{2}e^{-(\eta/2)^2})+$$

$$\left.\left.+\tfrac{1}{2}\sqrt{\frac{\pi}{2}}\,\eta\,\mathrm{erf}\left(\sqrt{2}\,\frac{\eta}{2}\right)-\frac{\pi}{32}(2-\eta^2).\,\mathrm{erf}^2\left(\frac{\eta}{2}\right)+\frac{\sqrt{\pi}}{2}\eta(1+\tfrac{1}{4}\cdot e^{-(\eta/2)^2})\cdot\mathrm{erf}\left(\frac{\eta}{2}\right)\right\}\right\}+$$

$$+(\hat{\varepsilon})^3\tfrac{1}{8}\sqrt{\frac{\pi}{3}}\,\eta\left[\mathrm{erf}\left(\frac{\eta}{2}\right)-\mathrm{erf}\left(\sqrt{3}\,\frac{\eta}{2}\right)\right]+O|(\hat{\varepsilon})^4\ln^2\hat{\varepsilon}|\,. \tag{3.26}$$

and,

$$\bar{v}=\frac{\hat{\varepsilon}}{2\sqrt{x}}\left\{-\eta e^{-(\eta/2)^2}+\hat{\varepsilon}e^{-(\eta/2)^2}\ \frac{\sqrt{\pi}}{2}\cdot\left(1-\frac{\eta^2}{2}\right)\mathrm{erf}\left(\frac{\eta}{2}\right)-\eta\cdot\right.$$

$$\cdot(1+\tfrac{1}{2}e^{-(\eta/2)^2})\Big]-\hat{\varepsilon}\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\sqrt{2}\,\frac{\eta}{2}\right)+(\hat{\varepsilon})^2\ln\hat{\varepsilon}\cdot\frac{\eta}{4\sqrt{3}}\cdot$$

$$\left.\cdot e^{-(\eta/2)^2}\left(3-\frac{\eta^2}{2}\right)\right\}+O|(\hat{\varepsilon})^3|\,. \tag{3.27}$$

For $\eta=0$, the first of the expressions above may be compared to Stewartson's equation, page 179 (op. cit. supra). It is seen that his constant β is equivalent to our G_2. It should again be noted that the term in v of order $(\bar{\varepsilon})^3$ no longer goes to zero as $\eta\to\infty$. The error is also symmetrical in η whereas v has to be antisymmetrical. The same effects, attributable to the application of the boundary-layer approximations, were noted by Goldstein, [1].

4. The Axisymmetrical Case

We consider here the equations (2.1) through (2.6) with $\delta=1$. The similarity transform and fundamental solution become, in analogy to (3.1) through (3.3),

$$\zeta=\frac{r}{2}x^{-\frac{1}{2}} \tag{4.1}$$

$$\psi_1(x,\zeta)=+\mathscr{B}f_1(\zeta) \tag{4.2}$$

i.e.

$$\bar{u}_1(x,\zeta)=-\frac{\mathscr{B}}{4x\zeta}f_1'(\zeta) \tag{4.3}$$

with

$$\mathscr{B}=\frac{C_D}{\pi}\cdot\frac{l\cdot U_{\mathrm{ref}}}{\nu}\,. \tag{4.4}$$

The linearized equation (2.5) to first order reduces in analogy to (3.12) to,

$$f_1'''+\frac{1}{\zeta}(2\zeta^2-1)f_1''+\left(2+\frac{1}{\zeta^2}\right)f_1'=0 \tag{4.5}$$

and, to first order in $\bar{u}$, equation (3.6) becomes for this case

$$\int_0^\infty f_1'(\zeta)\mathrm{d}\zeta = -1 . \tag{4.6}$$

The solution for f_1 is well known (c.f. Rosenhead, [15], p. 456)*

$$f_1' = -2\zeta \exp(-\zeta^2) \tag{4.7}$$

$$f_1 = \exp(-\zeta^2) - 1 . \tag{4.8}$$

We now propose to obtain higher approximations to the solution for the velocity profile, which will enable the calculation for the case in which $\bar{u}$ is not infinitely small. Introduce $\bar{u}$ at $\zeta = 0$ as a small perturbation parameter, thus

$$\varepsilon = \frac{\mathscr{B}}{2x} \tag{4.9}$$

and the expansion for the stream function becomes,

$$\Psi(x, \zeta) = +2x \sum_{k=1}^{\infty} \gamma_k(\varepsilon) f_k(\zeta) \tag{4.10}$$

$$\bar{u}(x, \zeta) = -\frac{1}{2\zeta} \cdot \sum_{k=1}^{\infty} \gamma_k(\varepsilon) f_k'(\zeta) \tag{4.11}$$

$$\gamma_1(\varepsilon) = \varepsilon . \tag{4.12}$$

The boundary conditions for the functions f_k are found by insertion into (2.2) to be as follows:

$$\left.\begin{aligned} (f_k'' - f_k'/\zeta)/\zeta &= 0 \quad \text{at} \quad \zeta = 0,\ x > 0 \\ f_k' - f_k/\zeta &= 0 \quad \text{at} \quad \zeta = 0,\ x > 0 \\ \text{and} \qquad & \\ f_k'/\zeta &= 0 \quad \text{at} \quad \zeta = \infty . \end{aligned}\right\} \tag{4.13}$$

We shall again introduce the Euler transformation. Then the expansion for the stream function becomes,

$$\Psi = 2x \cdot [\hat{\varepsilon} f_1 + (\hat{\varepsilon})^2 \cdot \ln \hat{\varepsilon} f_2 + (\hat{\varepsilon})^2 \cdot f_3 + (\hat{\varepsilon})^3 \ln^2 \hat{\varepsilon} f_4 + (\hat{\varepsilon})^3 \ln \hat{\varepsilon} f_5 + (\hat{\varepsilon})^3 f_6 + \ldots] . \tag{4.14}$$

Inserting into equations (2.5) and collecting terms of equal order in $(\hat{\varepsilon})$, the following set of confluent hypergeometric equations is obtained for the functions f_k:

$$f_2''' + \left(2\zeta - \frac{1}{\zeta}\right) f_2'' + \left(6 + \frac{1}{\zeta^2}\right) f_2' = 0 \tag{4.15}$$

$$f_3''' + \left(2\zeta - \frac{1}{\zeta}\right) f_3'' + \left(6 + \frac{1}{\zeta^2}\right) f_3' = -4f_2' - \frac{2}{\zeta}(f_1')^2 + 4f_1' \tag{4.16}$$

$$f_4''' + \left(2\zeta - \frac{1}{\zeta}\right) f_4'' + \left(10 + \frac{1}{\zeta^2}\right) f_4' = 0 \tag{4.17}$$

$$f_5''' + \left(2\zeta - \frac{1}{\zeta}\right) f_5'' + \left(10 + \frac{1}{\zeta^2}\right) f_5' = -8f_4' - 2\left(\frac{3}{\zeta} f_1' - 4\right) f_2' + \frac{2}{\zeta}\left(f_1'' - \frac{f_1'}{\zeta}\right) f_2 . \tag{4.18}$$

The solutions can be obtained in straightforward manner as follows:

$$f_2' = C_1 \zeta(\zeta^2 - 1)\exp(-\zeta^2) \tag{4.19}$$

$$f_2 = -\frac{C_1}{2} \zeta^2 \exp(-\zeta^2) \tag{4.20}$$

$$f_3' = \zeta e^{-\zeta^2} \cdot \{(\zeta^2 - 1) \cdot [-\tfrac{1}{2} Ei(-\zeta^2) + 2C_1 \ln \zeta + C_2] - \tfrac{1}{2} e^{-\zeta^2} - 2(1 + C_1)\} \tag{4.21}$$

$$f_3 = \tfrac{1}{4} e^{-\zeta^2} \{\zeta^2 [+Ei(-\zeta^2) - 4C_1 \ln \zeta - 2C_2] + e^{-\zeta^2} + 2(C_1 + 2)\} + C_3 \tag{4.22}$$

* Rosenhead's $f = \frac{1}{\zeta} \frac{df_1}{d\zeta}$,

$$f_4' = +C_4\zeta e^{-\zeta^2}(1-2\zeta^2+\tfrac{1}{2}\zeta^4) \tag{4.23}$$

$$f_4 = +\frac{C_4}{2}\cdot e^{-\zeta^2}\cdot\zeta^2(1-\tfrac{1}{2}\zeta^2)+C_5 \tag{4.24}$$

$$f_5' = \zeta e^{-\zeta^2}\{(1-2\zeta^2+\tfrac{1}{2}\zeta^4)\cdot[+\tfrac{1}{4}Ei(-\zeta^2)+4C_4\ln\zeta+C_6]+ \\ +(\tfrac{1}{8}e^{-\zeta^2}+1)\cdot(\zeta^2-1)+2C_4(3-2\zeta^2)\} \tag{4.25}$$

$$f_5 = \tfrac{1}{8}e^{-\zeta^2}\left\{\zeta^2\left(1-\frac{\zeta^2}{2}\right)[Ei(-\zeta^2)-2\ln\zeta+4C_6]+\frac{e^{-\zeta^2}}{2}(1-\zeta^2)+\tfrac{1}{2}(1-11\zeta^2)\right\}+C_7, \tag{4.26}$$

where Ei is the exponential integral. The manner in which these solutions have been found is essentially that of Rotem and Wygnanski [8]. Indeed, for $k=1, 2$ the equations and their solutions happen to be identical with the first two terms calculated in [8] for the case of a weak axisymmetrical jet.

The constants C_1 through C_7 appearing in the equations above have now to be determined. In the case of the *two-dimensional* wake [2] and the two-dimensional weak jet [8] these constants are obtained by a technique of "switchback" formalized recently by Chang [4], in conjunction with the condition that the vorticity, and hence also the velocity in the direction of streaming, must decay at a rate faster than algebraical with the distance normal to the plane of symmetry of the wake provided the outer flow in potential. In the *present case* the condition of exponential decay normal to the axis of symmetry is fulfilled automatically: all constants except C_2 and C_6 are determined by the boundary conditions. However, these two constants remain entirely arbitrary. It may be presumed from the properties of the basic parabolic partial differential equations, which are an approximation to the complete elliptic equations of flow, see [2] that their determination depends upon the form of the initial velocity profile upstream.

The other constants have the following values: $C_1=\frac{1}{2}$, $C_3=-\frac{3}{2}$, $C_4=-\frac{1}{8}$, $C_5=0$, $C_7=-\frac{1}{8}$. As far as *numerical* evaluation is concerned the expansion will have to be terminated at the second term due to the unavailability of the constant C_2.

We are now in a position to write down the expression for $\bar{u}$ and for v as follows:

$$\bar{u} = \frac{\hat{\varepsilon}}{2}e^{-\zeta^2}\Big\{+2-\hat{\varepsilon}\ln\hat{\varepsilon}\cdot\tfrac{1}{2}(\zeta^2-1)+\hat{\varepsilon}\{(\zeta^2-1)[+\tfrac{1}{2}Ei(-\zeta^2)-\ln\zeta-C_2]+\tfrac{1}{2}e^{-\zeta^2}+3\} \\ +\tfrac{1}{8}(\hat{\varepsilon})^2\ln^2(\hat{\varepsilon})(1-2\zeta^2+\tfrac{1}{2}\zeta^4)-(\hat{\varepsilon})^2\ln\hat{\varepsilon}\{(1-2\zeta^2+\tfrac{1}{2}\zeta^4)\cdot[\tfrac{1}{4}Ei(-\zeta^2)-\tfrac{1}{2}\ln\zeta+C_6]+ \\ +(\tfrac{1}{8}e^{-\zeta^2}+1)\cdot(\zeta^2-1)-\tfrac{1}{4}(3-2\zeta^2)\}\Big\}+O|(\hat{\varepsilon})^3| \tag{4.27}$$

and,

$$\bar{v} = -\frac{\hat{\varepsilon}}{4}\frac{1}{x^{\frac{1}{2}}\zeta}\Big\{4\zeta^2e^{-\zeta^2}+\hat{\varepsilon}\ln\hat{\varepsilon}[\zeta^2(2-\zeta^2)e^{-\zeta^2}] \\ -\hat{\varepsilon}\cdot e^{-\zeta^2}\{\zeta^2\cdot(2-\zeta^2)\cdot[Ei(-\zeta^2)-2\ln\zeta-2C_2]+(1-\zeta^2)e^{-\zeta^2}+(1-7\zeta^2)\}+ \\ -10\hat{\varepsilon}-(\hat{\varepsilon})^2\ln^2(\hat{\varepsilon})\left(\frac{\zeta}{2}\right)^2e^{-\zeta^2}(3\zeta^2-\tfrac{1}{2}\zeta^4-3)-(\hat{\varepsilon})^2\ln\hat{\varepsilon}\ \tfrac{1}{2}e^{-\zeta^2}\cdot \\ \cdot\{\zeta^2(3-3\zeta^2+\tfrac{1}{2}\zeta^4)\cdot[Ei(-\zeta^2)-2\ln\zeta+4C_6]+(1-\tfrac{3}{2}\zeta^2+\tfrac{1}{2}\zeta^4)\cdot \\ \cdot e^{-\zeta^2}+(1-15\zeta^2+\tfrac{13}{2}\zeta^4)\}+(\hat{\varepsilon})^2\ln\hat{\varepsilon}\Big\}+O|(\hat{\varepsilon})^3|\,. \tag{4.28}$$

It should be realized that v as per equation (4.28) may not vanish with $\zeta\to\infty$. This blemish is a result of the application of the boundary-layer approximations to the problem on hand, [1]. However, any remainder must be of order $(\hat{\varepsilon})^3$ or smaller.

5. Results and Discussion

Higher order approximations to the laminar boundary-layer flow of an incompressible fluid in the wake of a symmetrical disturbance are given in the present paper. The two-dimensional case, documented elsewhere in great detail, [1], [2], [3] and [4] is reconsidered. The Euler transformation is introduced and higher-order expansion terms are derived. The asymptotic expansions given in the paper are, of course, valid only to the extent that the boundary layer approximations apply, i.e. (for the rotationally symmetrical case) within a space of revolution with the centre of the wake as axis of symmetry. The terms neglected in the complete equations of motion become of order unity for very small x, where the expansion is not applicable in any case (as ε becomes large), and at very large ζ (respectively η), x being given.

The axisymmetrical case is expanded in a like manner, but in both cases the inner and outer coördinate expansion problem of matching with the near wake, considered by Meksyn [16] and Berger [11] is not treated: this, mainly, because its detailed form would depend on the particular upstream conditions obtaining, a subject which is outside the scope of the present work.

In order to improve the convergence of the asymptotic expansions the *Euler* transformation was introduced, both in the axisymmetrical and in the two-dimensional configuration. This modification has, of course, no effect upon the first order term. The *form* of the sequence is not affected by the introduction of the *Euler* transformation either, though the detailed functions certainly are. The convergence of an asymptotic sequence of a limited number of terms should thereby be improved.

An interesting feature of the axisymmetrical wake is that the presence of terms of logarithmic order in the expansion parameter is required in order to satisfy the *boundary* conditions, rather than to ensure the exponential decay of vorticity as in the case of two-dimensional wake flow. In the present investigation the terms of logarithmic order were, of course, assumed *a priori*, and their justification was then shown *a posteriori*. The form of the expansion assumed in the present work appears to be entirely self consistent, with no other terms needed. Nevertheless, uniqueness is not one of the features of this type of expansion.

It is interesting to note that in the two-dimensional wake the normal component of velocity does not vanish as $\eta \to \infty$ the remainder being of order $(\hat{\varepsilon})^3$. In the axisymmetrical case also v vanish as $\zeta \to \infty$. However, starting with order $(\hat{\varepsilon})^2$, the decrease in v with ζ is algebraical and not exponential. It is conceivable that the choice of optimal coordinates in the sense of Kaplun would have postponed this particular difficulty.

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